Annals of Nuclear Energy 97 (2016) 47–52

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

A nonstiff solution for the stochastic neutron point kinetics equations

M. Wollmann da Silva [a], R. Vasques [b,*], B.E.J. Bodmann [a], M.T. Vilhena [a]

[a] UFRGS – Federal University of Rio Grande do Sul, Av. Osvaldo Aranha 99, 90046-900 Porto Alegre, RS, Brazil
[b] University of California, Berkeley, Department of Nuclear Engineering, 4155 Etcheverry Hall, Berkeley, CA 94720-1730, United States

ARTICLE INFO

Article history:
Received 29 February 2016
Received in revised form 21 June 2016
Accepted 23 June 2016

Keywords:
Point kinetics
Stochastic moments
Stiffness
Decomposition method

ABSTRACT

We propose an approach to solve the stochastic neutron point kinetics equations using an adaptation of the diagonalization-decomposition method (DDM). This new approach (Double-DDM) yields a nonstiff solution for the stochastic formulation, allowing the calculation of the neutron and precursor densities at any time of interest without the need of using progressive time steps. We use Double-DDM to compute results for stochastic problems with constant, linear, and sinusoidal reactivities. We show that these results strongly agree with those obtained by other approaches established in the literature. We also compute and analyze the first four statistical moments of the solutions.

1. Introduction

The neutron point kinetics equations (Hetrick, 1971; Kinard and Allen, 2004; Hayes and Allen, 2005) are the coupled differential equations for the neutron density and for the delayed neutron precursor concentrations. These equations model the time-dependent behavior of a nuclear reactor and provide insight into the dynamics of its operation. The time-dependent parameters in this system are the reactivity function and the neutron source term.

The neutron density and delayed neutron precursor concentrations vary randomly with time; however, the point kinetics equations are deterministic and can only be used to estimate average values. Random fluctuations in the neutron density and precursor concentrations can be significant at low power levels (Hurwitz et al., 1963), which points to the importance of estimating these variations.

Hayes and Allen (2005) have generalized the standard deterministic point kinetics equations, deriving a system of stochastic differential equations that model the random behavior of the neutron density and the precursor concentrations in a point reactor. Due to the issue of stiffness, this system was implemented numerically using a stochastic piecewise constant approximation method (Stochastic PCA). Work performed by Saha Ray (2012) has shown that order 1.5 strong Taylor and Euler–Maruyama numerical methods are valid computational alternatives to Stochastic PCA in solving the stochastic point kinetics equations. However, with the exception of cases modeled with either none or only one group of neutron precursors, the stiffness of the problem remains.

In this paper we propose to solve this stochastic formulation using a double decomposition approach based on the diagonalization-decomposition method (DDM) decribed by Wollmann da Silva et al. (2014). This proposed method is the major novelty and principal contribution of this work, yielding a *nonstiff* solution for the stochastic point kinetics equations. Specifically, this approach allows the calculation of the neutron and precursor densities at any time of interest without the need of using progressive time steps. This solution is obtained with a minimal amount of numerical approximations of the model; the largest numerical effort lies in the truncation of the decomposition and the integrations required by DDM.

The major caveat in this approach is that convergence of DDM is yet to be proven. For this reason, a Lyapunov criterion (Boichenko et al., 2005) is used to guarantee convergence (cf. Petersen et al., 2011; Wollmann da Silva et al., 2014). We present computational results for problems with constant, linear, and sinusoidal reactivities. The results of the proposed method are compared against those of other approaches established in the literature, showing strong agreement. We also compute the first four statistical moments of the solutions.

This work is an expanded version of a recent conference paper (Wollmann da Silva et al., 2015). The remainder of this paper is organized as follows. In Section 2 we present a brief review on the key aspects of DDM. In Section 3 we formulate the stochastic

* Corresponding author at: University of California, Berkeley, Department of Nuclear Engineering, 4103 Etcheverry Hall, Berkeley, CA 94720-1730, United States.
E-mail addresses: milena.wollmann@ufrgs.br (M. Wollmann da Silva), richard.vasques@fulbrightmail.org (R. Vasques), bardobodmann@ufrgs.br (B.E.J. Bodmann), vilhena@mat.ufrgs.br (M.T. Vilhena).

http://dx.doi.org/10.1016/j.anucene.2016.06.026
0306-4549/

point kinetics equations. We introduce the proposed double decomposition approach in Section 4. Numerical results are given in Section 5 for problems with constant (Section 5.1) and time-dependent (Section 5.2) reactivities. The paper concludes in Section 6 with a discussion of the work presented.

2. The diagonalization-decomposition method (DDM)

Following the work performed by Wollmann da Silva et al. (2014), one can obtain an analytical representation for the solution of the neutron point kinetics equations that is free of stiffness. The neutron point kinetics equations with six groups of precursors and time-dependent reactivity $\rho(t)$ are written as:

$$\frac{d}{dt}n(t) = \frac{\rho(t)-\beta}{\Lambda}n(t) + \sum_{i=1}^{6}\lambda_i C_i(t), \quad n(0) = n_0, \tag{2.1a}$$

$$\frac{d}{dt}C_i(t) = \frac{\beta_i}{\Lambda}n(t) - \lambda_i C_i(t), \quad C_i(0) = \frac{\beta_i n_0}{\Lambda\lambda_i}, \tag{2.1b}$$

for $i = 1, 2, \ldots, 6$. Here, $n(t)$ is the neutron density; $C_i(t)$ is the density of the ith delayed neutron precursor group; λ_i is the decay constant for a specific group i; Λ represents the neutron mean generation time; and β_i represents the delayed-neutron fraction in a specific group i. The total fraction of delayed neutrons is given by $\beta = \sum_{i=1}^{6}\beta_i$.

A recursive scheme with finite recursive depth R is used to obtain a solution. The truncation index is determined with exponential convergence by the Lyapunov criterion (Boichenko et al., 2005; Petersen et al., 2011), evaluated after each recursion step. The neutron population and the precursors concentrations are written in terms of the solution from a recursion initialization ($j = 0$) and the respective correction terms ($j > 0$) for an appropriate $R \in \mathbb{N}$:

$$n(t) = \sum_{j=0}^{R} n_j(t), \tag{2.2a}$$

$$C_i(t) = \sum_{j=0}^{R} C_{ij}(t). \tag{2.2b}$$

The combination of Eqs. (2.1) and (2.2) yields a system with $7 \times R$ unknowns. We define

$$\boldsymbol{Y}(t) = \sum_{j=1}^{R}\boldsymbol{Y}_j(t), \tag{2.3a}$$

$$\boldsymbol{Y}_j(t) = [n_j(t), C_{1j}(t), C_{2j}(t), C_{3j}(t), C_{4j}(t), C_{5j}(t), C_{6j}(t)]^T, \tag{2.3b}$$

$$\boldsymbol{\Omega} = \mathrm{diag}\left(\frac{\rho_0-\beta}{\Lambda}, -\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5, -\lambda_6\right), \tag{2.3c}$$

and

$$\boldsymbol{\Xi} = \begin{bmatrix} \frac{\rho_1(t)}{\Lambda} & \lambda_1 & \lambda_2 & \ldots & \lambda_6 \\ \frac{\beta_1}{\Lambda} & 0 & 0 & \ldots & 0 \\ \frac{\beta_2}{\Lambda} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\beta_6}{\Lambda} & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{2.3d}$$

where the constant ρ_0 and $\rho_1(t)$ are such that $\rho(t) = \rho_0 + \rho_1(t)$. Given the recursive system

$$\frac{d}{dt}\boldsymbol{Y}_0(t) - \boldsymbol{\Omega}\boldsymbol{Y}_0(t) = 0, \tag{2.4a}$$

$$\frac{d}{dt}\boldsymbol{Y}_j(t) - \boldsymbol{\Omega}\boldsymbol{Y}_j(t) = \boldsymbol{\Xi}(t)\boldsymbol{Y}_{j-1}(t), \qquad j > 0\,, \tag{2.4b}$$

the solution of Eq. (2.4a) is

$$\boldsymbol{Y}_0(t) = \exp(\boldsymbol{\Omega}t)\boldsymbol{Y}_0(0), \tag{2.5a}$$

with $\boldsymbol{Y}_0(0) = [n_0, C_1(0), C_2(0), \ldots, C_6(0)]^T$. Equation (2.4b) may be formally solved by the Laplace transform:

$$\boldsymbol{Y}_j(t) = \exp(\Omega t)\cancelto{0}{\boldsymbol{Y}_j(0)} + \int_0^t \exp(\boldsymbol{\Omega}\tau)\boldsymbol{\Xi}(t-\tau)\boldsymbol{Y}_{j-1}(t-\tau)d\tau, \qquad j > 0, \tag{2.5b}$$

since the initial condition from Eqs. (2.1) is fully absorbed in Eq. (2.5a). The integral in Eq. (2.5b) is evaluated using the Gauss–Legendre method.

A flowchart describing the implementation of this method is given in Fig. 1. The solution is obtained in an analytical representation that may be evaluated for any time value (free of stiffness).

3. The stochastic formulation

Hayes and Allen (2005) derived a system of Itô stochastic differential equations that model the dynamics of the neutron density and the delayed neutron precursors in a nuclear reactor. This formulation describes the variation of the population and can be interpreted as a balance between deaths, births, and transformations of neutrons in the system. The probabilities of these events are determined by the physical parameters of the model, such as the total and partial delayed neutron fractions; the fraction of delayed neutrons of each precursor group; the decay constant of each group; and the average number of neutrons produced in each fission.

Assuming a time interval small enough such that only one event occurs, one can write

$$\frac{d}{dt}\boldsymbol{Y}(t) = \widehat{\boldsymbol{A}}(t)\boldsymbol{Y}(t) + \boldsymbol{Q}(t) + \widehat{\boldsymbol{B}}^{\frac{1}{2}}(t)\frac{d}{dt}\boldsymbol{W}(t), \tag{3.1a}$$

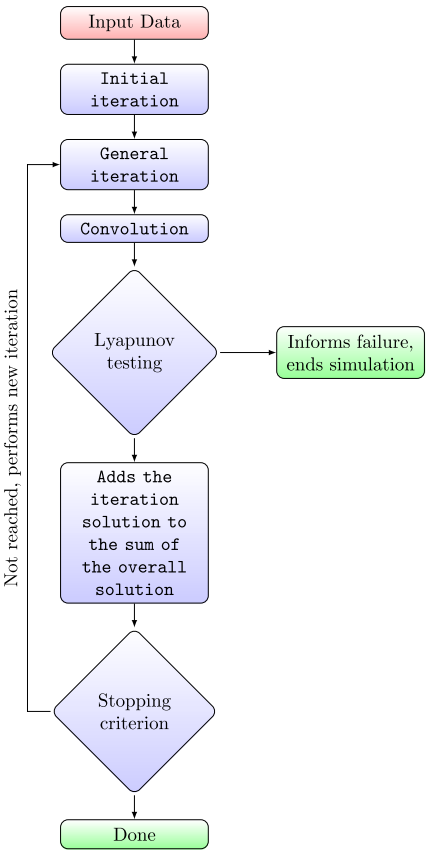

Fig. 1. DDM approach to solve the deterministic problem.

where

$$\boldsymbol{Y}(t) = [n(t), C_1(t), C_2(t), C_3(t), C_4(t), C_5(t), C_6(t)]^T, \tag{3.1b}$$

$$\boldsymbol{Q}(t) = [q(t), 0, 0, 0, 0, 0, 0]^T, \tag{3.1c}$$

$$\boldsymbol{W}(t) = [W_0(t), W_1(t), W_2(t), W_3(t), W_4(t), W_5(t), W_6(t)]^T, \tag{3.1d}$$

$$\widehat{\boldsymbol{A}}(t) = \boldsymbol{\Omega} + \boldsymbol{\Xi}(t), \tag{3.1e}$$

$$\widehat{\boldsymbol{B}}(t) = \begin{bmatrix} \zeta & a_1 & a_2 & \dots & a_6 \\ a_1 & b_{1,1} & b_{1,2} & \dots & b_{1,6} \\ a_2 & b_{2,1} & b_{2,2} & \dots & b_{2,6} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_6 & b_{6,1} & \dots & b_{6,5} & b_{6,6} \end{bmatrix}. \tag{3.1f}$$

Here, $\boldsymbol{\Omega}$ and $\boldsymbol{\Xi}(t)$ are given by Eqs. (2.3c) and (2.3d), $W_i(t)$ are Wiener processes, and

$$\zeta = \left(\frac{2\beta - 1 - \rho(t) + (1-\beta)^2 v}{\Lambda}\right) n(t) \sum_{i=1}^{6} \lambda_i C_i(t) + q(t), \tag{3.1g}$$

$$a_i = \frac{\beta_i}{\Lambda}(-1 + (1-\beta)v) n(t) - \lambda_i C_i(t), \tag{3.1h}$$

$$b_{i,j} = \frac{\beta_i \beta_j v}{\Lambda} n(t) + \delta_{ij} \lambda_i C_i(t), \tag{3.1i}$$

with v = total number of neutrons per fission (prompt neutrons + delayed neutrons). Note that if $\widehat{\boldsymbol{B}}(t) = 0$, then Eq. (3.1a) (with $\boldsymbol{Q}(t) = 0$) reduces to the deterministic problem discussed in Section 2.

4. The proposed method (Double-DDM)

We propose to solve the stochastic formulation in Section 3 by adapting the recursive method described in Section 2. This approach yields a nonstiff solution to the stochastic system in Eqs. (3.1).

Since the matrix $\widehat{\boldsymbol{B}}$ depends on both the neutron populations and the delayed neutron concentrations, we resort to a double decomposition to obtain a solution for this problem:

I. Using DDM as described in Section 2, Eq. (3.1a) is solved for $\widehat{\boldsymbol{B}} = 0$; this yields the deterministic solution $\boldsymbol{Y}(t) = \boldsymbol{Y}_d(t)$;
II. $\boldsymbol{Y}_d(t)$ is used to determine the matrix $\widehat{\boldsymbol{B}}$ for a sequence of discrete time steps (its components are constant in each time step);
III. $\widehat{\boldsymbol{B}}^{\frac{1}{2}}$ is obtained through diagonalization ($\widehat{\boldsymbol{B}}$ is symmetric);
IV. Since $\boldsymbol{Q}(t)$ is known for a specific time interval, a decomposition scheme analogue to DDM is applied:

$$\frac{d}{dt}\boldsymbol{Y}_0(t) - \boldsymbol{\Omega}\boldsymbol{Y}_0(t) = 0, \tag{4.1a}$$

$$\frac{d}{dt}\boldsymbol{Y}_j(t) - \boldsymbol{\Omega}\boldsymbol{Y}_j(t) = \boldsymbol{\Xi}(t)\boldsymbol{Y}_{j-1}(t) + \mathscr{F}(t), \tag{4.1b}$$

where $\mathscr{F}(t) = \boldsymbol{Q}(t) + \widehat{\boldsymbol{B}}^{\frac{1}{2}} \frac{d}{dt}\boldsymbol{W}(t)$ are constants known in each time step.

The total number K of stochastic components "drawn" in this approach is defined by the Central Limit Theorem (Meyer, 1965) to guarantee a small statistical error.

For example, let $\overline{\boldsymbol{Y}}$ be the "real" expected value of $\boldsymbol{Y}$. The *numerical* expected value $E(\boldsymbol{Y})$ and variance $\mathrm{Var}(\boldsymbol{Y})$ are given by

$$E(\boldsymbol{Y}) = \sum_{k=1}^{K} \frac{\boldsymbol{Y}^{(k)}}{K}, \tag{4.2a}$$

$$\mathrm{Var}(\boldsymbol{Y}) = \sigma^2(\boldsymbol{Y}) = \sum_{k=1}^{K} \frac{\left(\boldsymbol{Y}^{(k)} - E(\boldsymbol{Y})\right)^2}{K}, \tag{4.2b}$$

where the index k represents the different choices of stochastic components (histories). The Central Limit Theorem (CLM) implies that the inequality

$$\left|1 - \frac{\overline{\boldsymbol{Y}}}{E(\boldsymbol{Y})}\right| < \frac{2\sigma(\boldsymbol{Y})}{\sqrt{K} E(\boldsymbol{Y})} \tag{4.3}$$

holds with probability 0.95.

For all results shown in this paper, K is large enough to guarantee a statistical error smaller than 2% with 95% confidence for the first two raw moments. In the example given above, this means that K is chosen such that the right-hand side of Eq. (4.3) is smaller than 0.02. The flowchart in Fig. 2 describes the implementation of the proposed method.

5. Numerical results

In this section we present numerical results for the Double-DDM approach proposed in Section 4 for examples with (i) constant and (ii) time-dependent reactivities. We compare these results against those obtained with other approaches established in the literature.

In addition to the expected value $E(\boldsymbol{Y})$ and variance $\mathrm{Var}(\boldsymbol{Y})$ defined in Eqs. (4.2), we also calculate two standardised moments for the neutron density n: *skewness* and *excess kurtosis*. These are defined as

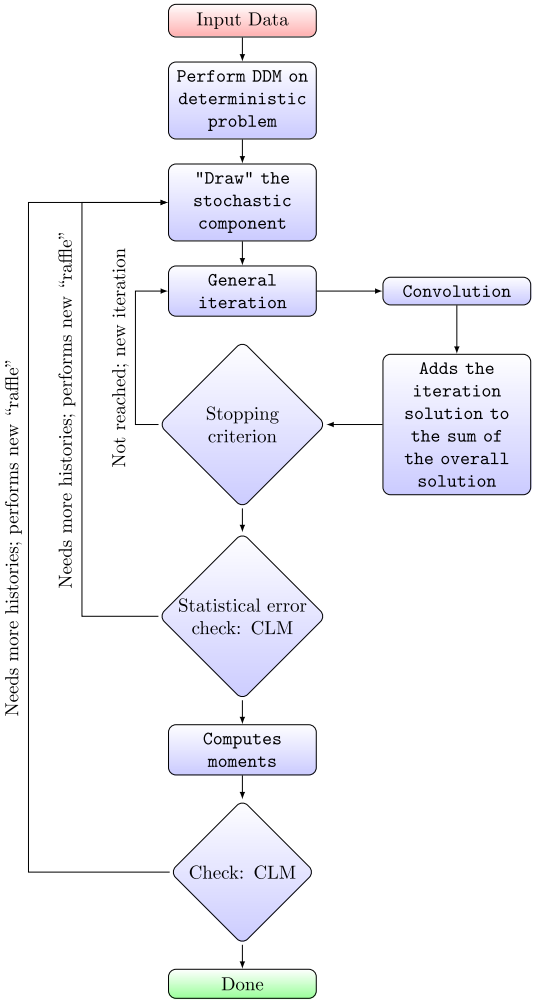

Fig. 2. Double-DDM approach to solve the stochastic problem.

$$\text{Skew}(n) = \gamma_1(n) = \sum_{k=1}^{K} \frac{\left(n^{(k)} - E(n)\right)^3}{K\sigma^3(n)}, \tag{5.1a}$$

$$\text{Kurt}(n) - 3 = \gamma_2(n) = \sum_{k=1}^{K} \left[\frac{\left(n^{(k)} - E(n)\right)^4}{K\sigma^4(n)}\right] - 3, \tag{5.1b}$$

which gives us further insight on the behavior of the stochastic solutions.

5.1. Constant reactivity

In the following examples we present the results of four methods established in the literature: Monte Carlo and Stochastic PCA (Hayes and Allen, 2005); order 1.5 strong Taylor and Euler–Maruyama (Saha Ray, 2012). These results are reproduced here as they were reported in the aforementioned references.

We compare these results with those obtained with the deterministic diagonalization-decomposition method (DDM) and with the Double-DDM approach. The solutions of the deterministic DDM are obtained by solving Eq. (3.1a) with $\widehat{\boldsymbol{B}} = 0$; we point out that this is also the first step of Double-DDM.

5.1.1. Step-reactivity insertion: one precursor

This example does not model an actual physical nuclear reactor problem. Nevertheless, considering only one group of precursors implies that the stiffness of the problem disappears; this provides a simple computational solution that is useful for a first comparison with other methods.

The physical parameters are $\lambda_1 = 0.1$, $\beta = \beta_1 = 0.05$, $v = 2.5$, $q = 200$, $\Lambda = \frac{2}{3}$, and $\rho(t) = -\frac{1}{3}$ for $t \geqslant 0$. The initial condition is $\boldsymbol{Y}(0) = [400,\ 300]^T$. The expected values and standard deviations of $n(t)$ and $C_1(t)$ at time $t = 2$ are presented in Table 1 for each of the methods. A total of $K = 1,000$ histories were accumulated for the Double-DDM approach. Skewness and excess kurtosis for the neutron density were found to be $\gamma_1(n(2)) = -1 \times 10^{-10}$ and $\gamma_2(n(2)) = 3 \times 10^{-11}$. The fact that γ_1 and γ_2 are nearly zero implies that the distribution of stochastic solutions is symmetric and has Gaussian-like peak and tail.

It can be seen that there exists a close agreement between Double-DDM and the results obtained with Monte Carlo and Stochastic PCA. Euler–Maruyama and order 1.5 strong Taylor (Taylor 1.5) yield slightly higher results, very close to those obtained with the deterministic DDM.

5.1.2. Step-reactivity insertion: six precursors

The following two examples model step-reactivity insertions in a nuclear reactor with ^{235}U as a fissile material. In this case, we consider a stiff system of equations with six precursor groups. The set of physical parameters is taken from Kinard and Allen (2004): $v = 2.5$, $\Lambda = 0.00002$, $q = 0$, and $\beta = 7 \times 10^{-3}$, with β_i and λ_i given in Table 2. The initial condition is given by

$$\boldsymbol{Y}(0) = 100\left[1, \frac{\beta_1}{\lambda_1\Lambda}, \frac{\beta_2}{\lambda_2\Lambda}, \frac{\beta_3}{\lambda_3\Lambda}, \frac{\beta_4}{\lambda_4\Lambda}, \frac{\beta_5}{\lambda_5\Lambda}, \frac{\beta_6}{\lambda_6\Lambda}\right]^T. \tag{5.2}$$

Table 1
Results for one group of precursors and step-reactivity insertion $\rho(t) = -1/3$.

	Monte Carlo	Stochastic PCA	Euler Maruyama	Taylor 1.5	DDM	Double DDM
$E(n(2))$	400.03	395.32	412.23	412.10	412.13	402.35
$\sigma(n(2))$	27.311	29.411	34.391	34.519	–	28.610
$E(C_1(2))$	300.00	300.67	315.96	315.93	315.93	305.84
$\sigma(C_1(2))$	7.8073	8.3564	8.2656	8.3158	–	7.9240

Table 2
Fraction of delayed neutrons and decay constants for the precursor groups.

Group	1	2	3	4	5	6
$\beta_i \times 10^{-3}$	0.266	1.491	1.316	2.849	0.896	0.182
λ_i	0.0127	0.0317	0.115	0.311	1.4	3.87

We compute results for a prompt subcritical step-reactivity insertion $\rho(t) = 0.003$ at time $t = 0.1$, and for a prompt critical step-reactivity insertion $\rho(t) = 0.007$ at time $t = 0.001$. We define

$$C(t) = \sum_{i=1}^{6} C_i(t), \tag{5.3}$$

and present the expected values and standard deviations for each of the methods in Table 3.

We collected $K = 10,000$ histories for the Double-DDM approach. For the subcritical step-reactivity $\rho(t) = 0.003$, skewness was found to be $\gamma_1(n(0.1)) = -1 \times 10^{-7}$ and excess kurtosis $\gamma_2(n(0.1)) = 1.3 \times 10^{-9}$. For the critical step-reactivity $\rho(t) = 0.007$, skewness and excess kurtosis were computed respectively as $\gamma_1(n(0.001)) = -1.02 \times 10^{-7}$ and $\gamma_2(n(0.001)) = 1.15 \times 10^{-8}$. These results for the third and fourth moments indicate that the distribution of the stochastic solutions is symmetric and has neither a sharp peak nor a heavy tail.

As in the previous example, the results obtained with Double-DDM are in close agreement to the results from Monte Carlo and Stochastic PCA. The results from Euler–Maruyama and order 1.5 strong Taylor are closer to those of deterministic DDM.

5.2. Time-dependent reactivity

The current literature lacks numerical results for the stochastic system in Eq. (3.1a) with time-dependent reactivities. For this reason, the results collected from the literature and reproduced next represent only the deterministic solution (with $\widehat{\boldsymbol{B}} = 0$). Although not ideal, this approach allows us to verify that the expected value obtained with Double-DDM closely agrees with well established models for problems with time-dependent reactivities. All the following examples take into account six precursor groups.

5.2.1. Linear reactivity $\rho(t) = at$

The following two examples model a ramp reactivity $\rho(t) = at$ for a thermal nuclear reactor. The physical parameters considered are: $\Lambda = 0.00001$, $n(0) = 1.0$, and $\beta = 6.403 \times 10^{-3}$, with β_i and λ_i taken from Lewins (1978) and given in Table 4.

We compute the neutron density $n(t)$ for two different choices of constant a: 0.25 and 0.5. The results obtained with Double-DDM are given in Table 5 for times t =0.25, 0.5, 0.75, and 1.0. We compare these results with those obtained with the Padé approximation (Aboanber and Nahla, 2002) and the generalization of the analytical exponential model (GAEM), as reported by Nahla (2008).

The number of histories collected for the case $a = 0.25$ was $K = 15,738$. We computed the higher moments for time $t = 1.0$, finding the standard deviation $\sigma(n(1)) = 0.978$, skewness $\gamma_1(n(1)) = -3.02 \times 10^{-7}$, and excess kurtosis $\gamma_2(n(1)) = -1 \times 10^{-7}$. For the case $a = 0.5$, we collected $K = 27,523$ histories, and found $\sigma(n(1)) = 1.11345$, $\gamma_1(n(1)) = -2.15 \times 10^{-6}$, $\gamma_2(n(1)) = -3 \times 10^{-8}$. This shows that, in both cases, the distribution of stochastic solutions is nearly normal.

Double-DDM shows good agreement with the other methods shown in Table 5. The results obtained with Double-DDM are slightly larger for the first case ($a = 0.25$), and slightly smaller for the second case. The relative differences are around 1% or smaller, being well within 1 standard deviation.

Table 3
Results for six groups of precursors with subcritical ($\rho(t) = 0.003$) and critical ($\rho(t) = 0.007$) step-reactivity insertions.

ρ		Monte Carlo	Stochastic PCA	Euler Maruyama	Taylor 1.5	DDM	Double DDM
0.003	$E(n(0.1))$	183.04	186.31	208.60	199.41	200.01	187.05
	$\sigma(n(0.1))$	168.79	164.16	255.95	168.55	–	167.83
	$E(C(0.1)) \times 10^5$	4.478	4.491	4.498	4.497	4.497	4.488
	$\sigma(C(0.1))$	1495.7	1917.2	1233.4	1218.8	–	1475.6
0.007	$E(n(0.001))$	135.67	134.55	139.57	139.57	139.61	135.86
	$\sigma(n(0.001))$	93.376	91.242	92.042	92.047	–	93.210
	$E(C(0.001)) \times 10^5$	4.464	4.464	4.463	4.463	4.463	4.463
	$\sigma(C(0.001))$	16.226	19.444	6.071	18.337	–	17.845

Table 4
Fraction of delayed neutrons and decay constants for the precursor groups.

Group	1	2	3	4	5	6
$\beta_i \times 10^{-3}$	0.246	1.363	1.203	2.605	0.819	0.167
λ_i	0.0127	0.0317	0.115	0.311	1.4	3.87

Table 5
Neutron density $n(t)$ with ramp reactivity $\rho(t) = at$.

a	Time	Padé	GAEM	DDM	Double DDM
0.25	0.25	1.069840	1.069541	1.069542	1.069763
	0.50	1.157065	1.156694	1.156695	1.157867
	0.75	1.265795	1.265331	1.265332	1.269374
	1.0	1.402562	1.401981	1.401982	1.403561
0.5	0.25	1.149544	1.149200	1.149210	1.137216
	0.50	1.369438	1.368927	1.368928	1.356934
	0.75	1.708411	1.707600	1.707601	1.695607
	1.0	2.276692	2.275271	2.275272	2.263278

5.2.2. Sinusoidal reactivity $\rho(t) = b\sin(t)$

The last example simulates a case with sinusoidal reactivity $\rho(t) = b\sin(t)$, with $b = 0.00073$, $\Lambda = 0.00003$, and $n(0) = 1.0$. The total fraction of delayed neutrons is given by $\beta = 6.473 \times 10^{-3}$, with β_i and λ_i shown in Table 6.

The results obtained with Double-DDM are presented in Table 7 for every whole second up until $t = 10$. We compare these results with those reported by Wollmann da Silva et al. (2014). These were obtained with the method introduced by Kang and Hansen (1973), referred to as K & H in Table 7, and with the method of Taylor series (cf. Nahla, 2011).

We collected $K = 2,934,237$ histories to achieve the requirement imposed for the statistical error. The higher moments for time $t = 10$ yield $\sigma(n(10)) = 1.3242$, $\gamma_1(n(10)) = -0.0048$, and $\gamma_2(n(10)) = -0.013$. These results indicate (i) a very small asymmetry in the distribution of stochastic solutions, with a slightly larger left tail; and (ii) a very small yet noticeable "flatter" peak when compared to a normal.

In general, Double-DDM closely agrees with the other methods presented here for comparison. Results displayed in Table 7 show that Double-DDM yields slightly smaller results than those attained with the other methods. This can be confirmed in Fig. 3,

Table 6
Fraction of delayed neutrons and decay constants for the precursor groups.

Group	1	2	3	4	5	6
$\beta_i \times 10^{-3}$	0.214	1.423	1.247	2.568	0.748	0.273
λ_i	0.0124	0.0305	0.111	0.301	1.14	3.01

Table 7
Neutron density $n(t)$ with sinusoidal reactivity $\rho(t) = 0.00073\sin(t)$.

Time	K & H	Taylor	DDM	Double DDM
0	1.00000	1.00000	1.00000	1.0000000
1	1.12397	1.12394	1.12396	1.1119659
2	1.16881	1.16884	1.16889	1.1568959
3	1.07443	1.07442	1.07448	1.0624859
4	0.95381	0.95380	0.95382	0.9418259
5	0.90737	0.90737	0.90735	0.8953559
6	0.96151	0.96158	0.96153	0.9495359
7	1.08748	1.08749	1.08745	1.0754559
8	1.17168	1.17164	1.17167	1.1596759
9	1.11128	1.11124	1.11130	1.0993059
10	0.98464	0.98464	0.98468	0.9726859

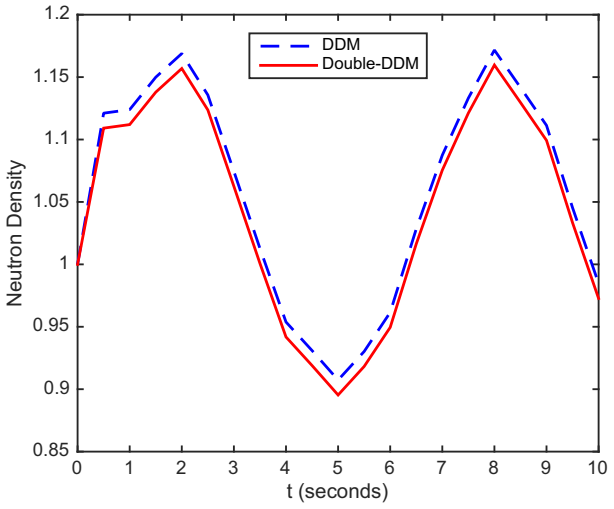

Fig. 3. Neutron density for a sinusoidal reactivity $\rho(t) = 0.00073\sin(t)$.

which depicts the average behavior of the stochastic solution compared to the deterministic solution (DDM). These differences are, once again, very small ($\approx 1\%$), and well within 1 standard deviation.

6. Conclusions

In this paper we propose an approach to solve the stochastic neutron point kinetics equations with a solution procedure that is free of stiffness. This is achieved through an adaptation of the diagonalization-decomposition method (DDM) introduced in Wollmann da Silva et al. (2014), wich provides a nonstiff solution for the deterministic point kinetics equations. DDM uses the

Laplace transform to obtain a formal solution, then applies a decomposition into a recursive scheme, using Gauss–Legendre integration.

The essential steps of the proposed approach (Double-DDM) are: (i) the deterministic problem is solved with DDM; (ii) the deterministic solution is used to build the stochastic component; (iii) another decomposition scheme analogue to DDM is used to solve the stochastic system. This allows the calculation of the neutron density and precursor concentrations at any time of interest, without the need to resort to progressive time steps. The elimination of stiffness comes from the fact that the evolution of the solution by recursion adds correction terms to the whole time interval of interest in each step, and simultaneously for each term that depends on a specific time scale.

Since convergence of DDM is yet to be proven, a Lyapunov criterion is used to guarantee convergence. The results of the proposed method are compared against results obtained through other approaches established in the literature. This comparison shows close agreement for problems with constant step-reactivity insertions, as well as time-dependent ramp reactivity and sinusoidal reactivity insertions.

In the current literature, the stochastic problem is mainly solved for constant reactivities; numerical solutions are limited to feasible time intervals due to the stiffness inherent to the problem. The mitigation of the stiffness character in solving the stochastic formulation is the major novelty and principal contribution of this work. Moreover, the analysis of the third and fourth moments of the stochastic solutions is, to the best of our knowledge, new.

The analyzed moments still depend on the size of the sample set and on the frequency with which the stochastic fluctuations are applied. In principle, an adjustment such as variance reduction and its consequences for higher moments could yield more realistic results. It would be necessary to find a reference scale in order to obtain such results independently of the sample size or frequency of application. This still needs to be identified, as well as a necessary ingredient to mimic reactor fluctuations. These tasks are left for future work.

Acknowledgments

Milena Wollmann da Silva would like to thank the Coordination for the Improvement of Higher Education Personnel (CAPES, Brazil) for financial support. Bardo E.J. Bodmann and Marco Túllio Vilhena thank the Federal University of Rio Grande do Sul (UFRGS) and also acknowledge financial support from the National Council of Scientific and Technological Development (CNPq, Brazil). Richard Vasques prepared this paper under award number NRC-HQ-84-14-G-0052 from the Nuclear Regulatory Commission. The statements, findings, conclusions, and recommendations are those of the authors and do not necessarily reflect the view of the U.S. Nuclear Regulatory Commission.

References

Aboanber, A.E., Nahla, A.A., 2002. Solution of the point kinetics equations in the presence of newtonian temperature feedback by padé approximations via the analytical inversion method. J. Phys. A: Math. Gen. 35 (45), 9609–9627.

Boichenko, V.A., Leonov, G.A., Reitmann, V., 2005. Dimension Theory for Ordinary Differential Equations. Teubner Verlag.

Hayes, J.G., Allen, E.J., 2005. Stochastic point-kinetics equations in nuclear reactor dynamics. Ann. Nucl. Energy 32 (6), 572–587.

Hetrick, D.L., 1971. Dynamics of Nuclear Reactors, first ed. University of Chicago Press, Chicago, U.S.A.

Hurwitz Jr, H., MacMillian, D.B., Smith, J.H., Storm, M.L., 1963. Kinetics of low source reactor startups part I. Nucl. Sci. Eng. 15 (2), 166–186.

Hurwitz Jr, H., MacMillian, D.B., Smith, J.H., Storm, M.L., 1963. Kinetics of low source reactor startups part II. Nucl. Sci. Eng. 15 (2), 187–196.

Kang, C.M., Hansen, K.F., 1973. Finite element methods for reactor analysis. Nucl. Sci. Eng. 51 (4), 456–495.

Kinard, M., Allen, E., 2004. Efficient numerical solution of the point kinetics equations in nuclear reactor dynamics. Ann. Nucl. Energy 31 (9), 1039–1051.

Lewins, J., 1978. Nuclear Reactor Kinetics and Control. Pergamon Press, Oxford, UK.

Meyer, P.L., 1965. Introductory Probability and Statistical Applications. Addison-Wesley, Massachusetts.

Nahla, A.A., 2008. Generalization of the analytical exponential model to solve the point kinetics equations of be- and d2o-moderated reactors. Nucl. Eng. Des. 238 (10), 2648–2653.

Nahla, A.A., 2011. Taylor series method for solving the nonlinear point kinetics equations. Nucl. Eng. Des. 241 (5), 1592–1595.

Petersen, C.Z., Dulla, S., Vilhena, M.T., Ravetto, P., 2011. An analytical solution of the point kinetics equations with time-variable reactivity by the decomposition method. Prog. Nucl. Energy 53 (8), 1091–1094.

Saha Ray, S., 2012. Numerical simulation of stochastic point kinetic equation in the dynamical system of nuclear reactor. Ann. Nucl. Energy 49, 154–159.

Wollmann da Silva, M., Leite, S.B., Vilhena, M.T., Bodmann, B.E.J., 2014. On an analytical representation for the solution of the neutron point kinetics equation free of stiffness. Ann. Nucl. Energy 71, 97–102.

Wollmann da Silva, M., Vilhena, M.T., Bodmann, B.E.J., Vasques, R., 2015. The solution of the neutron point kinetics equation with stochastic extension: an analysis of two moments. Proc. International Nuclear Atlantic Conference.